Work quickly and carefully, following directions closely. Answer all questions completely. You may use your own notes and calculators or Maple, but no books, people, or other resources. You have two and one half hours to complete the exam.

§I. TRUE and/or FALSE. Read each statement carefully. Circle your choice and briefly justify your answer.

1. TRUE or FALSE: There exists a function \( f \) such that \( f \) is continuous only on the irrational real numbers.

2. TRUE or FALSE: There exists a function \( f \) such that \( f \) is continuous on \( \mathbb{R} \), but \( f \) is not differentiable on \( \mathbb{R} \).

3. TRUE or FALSE: There exists a function \( f \) such that \( f \) is differentiable on \( \mathbb{R} \), but \( f \) is not continuous on \( \mathbb{R} \).

4. TRUE or FALSE: If \( f_n \) is continuous for all \( n \geq 0 \), then \( \int_a^b \lim_{k \to \infty} f_n(x) \, dx = \lim_{k \to \infty} \int_a^b f_n(x) \, dx \).

§II. MULTIPLE CHOICE. Read each statement carefully. Circle your choice.

1. Euler proved that the series \( \sum_{k=1}^{\infty} \frac{1}{k^2} \) converges to

   (a) \( \frac{\pi^2}{6} \)  
   (b) 1  
   (c) nothing since it diverges.  
   (d) none of the above  
   (e) all of the above

2. Suppose that \( f \) is continuous on the open interval \( (a, b) \) with \( -\infty < a < b < +\infty \). Then for any closed subinterval \( [c, d] \subset (a, b) \),

   (a) \( f \) is differentiable on \( [c, d] \).
   (b) \( f \) is uniformly continuous on \( [c, d] \).
   (c) \( f \) is invertible on \( [c, d] \).
   (d) None of the above.
   (e) All of the above.

3. The set of limit points \( S' \) of the set \( S = (-1, 1) \cup \{3, 4\} \) is

   (a) \( S' = \{-1, 1\} \)  
   (b) \( S' = \emptyset \)  
   (c) \( S' = [-1, 1] \)  
   (d) none of the above  
   (e) all of the above
III. Problems.

1. Since \( g(x) = e^{x^2} \) is continuous everywhere, then \( \int_{0}^{1} e^{x^2} \, dx \) exists and can be found using Riemann sums. Explain why the integral cannot be calculated using the Fundamental Theorem of Calculus.

2. Define \( \phi(x) = \begin{cases} \sin(x) & x \neq 0 \\ 0 & x = 0 \end{cases} \) on the open interval \((-a, a)\) where \(a > 0\).
   
   (a) Is \( \phi \) continuous on \((-a, a)\)?
   
   (b) Is \( \phi \) uniformly continuous on \((-a, a)\)? (Hint: What can you say about the derivative \( \phi' \)?)

3. Consider \( \sum_{k=1}^{\infty} \frac{x^k}{k} \) on the open interval \((-1, 1)\).
   
   (a) Does the series converge on \((-1, 1)\)?
   
   (b) Does the series converge uniformly on \((-1, 1)\)?

4. What is the Lebesgue measure of the set \( T = \left\{ \frac{n+1}{n} \mid n \in \mathbb{N} \right\} \)?

5. Write a brief reflection on the relation of the study of analysis to teaching elementary calculus.

IV. Proofs.

1. Use induction to prove: \( \sum_{k=1}^{n} k = \frac{n(n+1)}{2} \) for \( n \in \mathbb{N} \).

2. Choose (a) OR (b) to prove. Let \( \alpha \) and \( \beta \) be the last two nonzero digits in your ASU email ID. \( \alpha = \_\_ \); \( \beta = \_\_ \).
   
   (a) Use an \( \epsilon \)-\( \delta \) proof for: \( \lim_{x \to \alpha} 2x^2 + 1 = 2\alpha^2 + 1 = \_\_\_\_\_\_\_ \).

   (b) Use an \( \epsilon \)-\( \delta \) proof for: \( \lim_{x \to \beta} U(x - \beta) \) does not exist where \( U(t) = \begin{cases} 1 & t > 0 \\ 0 & t < 0 \end{cases} \).