Work quickly and carefully, following directions closely. Answer all questions completely.

§I. PROBLEMS.

1. Let $C$ be the astroid given by $f(t) = [\cos^3(t), \sin^3(t)]$ for $t \in [0, 2\pi]$. Let $P(t)$ be a point on $C$. Let $P_x$ and $P_y$ be the $x$- and $y$-intercepts of the line tangent to $C$ at $P(t)$. Show that the line segment $P_xP_y$ has constant length; i.e., the length of the segment is independent of $t$. (Click on the image to see a larger graph.)

2. Let $\mathbf{r}: \mathbb{R} \to \mathbb{R}^3$ be a vector-valued function that has 2 continuous derivatives for all $t$. Prove or disprove

$$\frac{d}{dt}[\mathbf{r}(t) \times \mathbf{r}'(t)] = \mathbf{r}(t) \times \mathbf{r}''(t).$$

3. Let $f(t) = \frac{2t^2}{1 + t^2}$ and set $C_{t\pi}$ to be the curve given by $[f(t) \cos(t), f(t) \sin(t)]$ for $t \in [0, 6\pi]$. Find the length of the curve $C$. Can you make a conjecture concerning the ratio $\frac{\text{length}(C_{2n\pi})}{4n}$ as $n \to \infty$?

4. Prove or disprove:
   Let $A_1 = B^2([1, 0], 1)$ and $A_{-1} = B^2([-1, 0], 1)$ be open balls in $\mathbb{R}^2$. Then $E = A_1 \cup A_{-1}$ is not separated.

5. Let $f(x, y) = \begin{cases} \frac{xy}{\sqrt{x^2 + y^2}} & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases}$. Determine $f_x$ and $f_y$. Is $f$ differentiable at $(0, 0)$?

6. A harmonic function is one that satisfies Laplace’s equation $\nabla^2 f(x, y) = 0$ where $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$.
   (a) Prove that the functions
      i. $f(x, y) = x^3 - 3xy^2$
      ii. $g(x, y) = 3x^2y - y^3$
   are harmonic.
   (b) Find $\frac{d^2z}{dt^2}$ for $z = x^3 - 3xy^2$ when $x(t) = \ln(t)$ and $y(t) = e^t$ without expanding $z$ in terms of $t$. 