Constructing a Nonmeasurable Set

Let \( X = [0, 1) \) and define the operation \( \oplus : X \rightarrow X \) by addition modulo 1. (Note: \( X \) is then equivalent to the unit circle via \( t \mapsto e^{2\pi it} \).) Let \( A \oplus x = \{ a \oplus x \mid a \in A \} \).

Define the relation \( \sim \) by

\[ x \sim y \text{ if and only if there is a rational } r \text{ such that } |x - y| = r \]

Exercises.

1. Show that for each rational \( r \), we have \( r \sim 0 \), and so all rationals are equivalent under \( \sim \).
2. Prove that \( \sim \) is an equivalence relation on \( X \).
3. Find \( [0] \).

Consider \( x_1 = \pi/10 \) and \( x_2 = \pi/30 \). Since \( x_1 - x_2 = \pi/15 \notin \mathbb{Q} \), then \( [x_1] \neq [x_2] \). Now consider \( x_3 = (\pi + 5)/10 \). Since \( x_1 - x_3 = 1/2 \in \mathbb{Q} \), we have that \( [x_1] = [x_3] \).

Since \( \sim \) is an equivalence relation, it partitions \( X \). Choose a representative \( h \) from each equivalence class in the partition of \( X \). (Axiom of Choice!) Gather these elements to form the set \( H \). Consider the collection of these sets \( \mathcal{H} = \{ H \oplus r \} \) where \( r \) ranges over the rationals in \( X \).

Exercises.

4. Determine whether \( H \) is countable or uncountable.
5. Verify that \( \mathcal{H} \) is a pairwise-disjoint family; i.e., \( (H \oplus r_1) \cap (H \oplus r_2) = \emptyset \) for \( r_1 \neq r_2 \).
6. Prove that \( X = \bigcup_{r \in \mathbb{Q} \cap X} (H + r) \).

Since Lebesgue measure is translation invariant, \( \mu(H \oplus r) = \mu(H) \) for all \( r \in \mathbb{Q} \cap X \). Assume that \( H \) is Lebesgue measurable with \( \mu(H) = \lambda \). Then, since \( \mathcal{H} \) is a countable family of disjoint sets,

\[
1 = \mu(X) = \mu \left( \bigcup_{r \in \mathbb{Q} \cap X} (H + r) \right) = \sum_{r \in \mathbb{Q} \cap X} \lambda
\]

We have a contradiction: If \( \lambda = 0 \), then \( 1 = 0 \). Otherwise, if \( \lambda > 0 \), then \( 1 = \infty \). Thus \( H \) cannot be Lebesgue measurable.

This construction is due to Giuseppe Vitali. (See also Vitali covering.)