**Proposition 1** (Sequence Ratio Test). Let \( \{a_n\} \) be a sequence with nonzero terms and set \( r = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| \).

- If \( r < 1 \), then \( a_n \to 0 \).
- If \( r = 1 \), the test fails.
- If \( r > 1 \), then \( a_n \) diverges and \( |a_n| \to \infty \).

**Example 1.** Give nontrivial examples of sequences \( \{a_n\} \)

1. with \( r < 1 \)

2. with \( r = 1 \) and

   (a) \( a_n \to 2 \)

   (b) \( a_n \to \infty \)

   (c) \( a_n \) oscillates and is bounded

   (d) \( a_n \) oscillates and is unbounded

3. with \( r > 1 \) and

   (a) \( a_n \to \infty \)

   (b) \( a_n \) diverges and \( |a_n| \to \infty \)
**Definition 1** (Rate of Convergence). Suppose that $a_n \to A$ and $b_n \to 0$. Then $a_n$ converges to $A$ with rate of convergence $b_n$ if there is a real constant $K$ such that

$$|a_n - A| \leq K \cdot |b_n|$$

eventually. Then

$$a_n = A + O(b_n).$$

**Example 2.** Give nontrivial examples or arguments justifying (not formal proofs) the following statements.

1. Show that $c_n = 2n/(n+1)$ converges to 2 with rate of convergence $1/n$. Thus $c_n = 2 + O(1/n)$.

2. Give an example of a sequence that
   - converges to 4
   - has rate of convergence $O(1/n^2)$

3. Let $d_n$ be given by

   $$d_n = \frac{3n^3 + 5n^2 + n + 1}{4n^3 + 3n^2 + 2n + 1}$$

   Find
   
   (a) $\lim_{n \to \infty} d_n$

   (b) the rate of convergence of $d_n$