2.4 Monotone Sequences

Definition 1. A sequence \( \{a_n\} \) is

1. increasing

2. eventually increasing

3. strictly increasing

4. eventually strictly increasing

- List three equivalent statements for: \( \{a_n\} \) is decreasing.

- Let \( c_n = \frac{(3n - 2)(n^2 + 1)}{n^3} \). Graph \( c_n \) v \( n \) and the range of \( c_n \).

Theorem 1. If \( \{a_n\} \) is an eventually increasing sequence, then either

1. \( \{a_n\} \) is bounded by some \( M \). Then there is an \( L \leq M \) such that \( \lim_{n \to \infty} a_n = L \).

or

2. \( \{a_n\} \) is unbounded. Then \( \lim_{n \to \infty} a_n = \infty \).
Proof. ‘Wolog’ let \( \{a_n\} \) be an increasing sequence. How is this without-loss-of-generality?

(a) Suppose that \( M \) is a bound for \( \{a_n\} \).
Let \( S = \{a_n \mid n \in \mathbb{N}\} \) be the range of \( a_n \). Then \( S \) is a set of real numbers bounded above by \( M \). Why?

Therefore \( S \) has a supremum. Why? Let \( \sup S = L \). Then \( L \leq M \).

Let \( \varepsilon > 0 \). (NTS: \( a_n \to L \); i.e., there is an \( n^* \) so that for any \( n > n^* \), we have \( |a_n - L| < \varepsilon \).)
Since \( L = \sup S \), there is an \( n^* \in \mathbb{N} \) such that

\[ a_{n^*} > L - \varepsilon. \]

Why?

If \( n > n^* \), then

\[ L - \varepsilon < a_{n^*} \leq a_n \leq L < L + \varepsilon. \]

Why?

That is, for any \( \varepsilon > 0 \) we can find an \( n^* \in \mathbb{N} \) such that if \( n > n^* \), then \( |a_n - L| < \varepsilon \). Thus \( a_n \) converges to \( L \).

(b) Suppose there is no upper bound for \( \{a_n\} \).
Exercice!

- Let \( 1 < p \in \mathbb{R} \). Determine whether the sequence \( c_n = n^p/p^n \) is eventually monotone.

- Investigate the sequence \( d_n = \sqrt{n^2 + n} - n \).
  1. Is \( d_n \) bounded? Above? Below?
  2. Is \( d_n \) increasing? Decreasing? Neither?
  3. Does \( d_n \) converge?