Georg Ferdinand Ludwig Philipp Cantor

Georg Cantor was born in Russia in 1845, but lived in Germany, except for beginning his university studies in Zurich, until his death in 1918. After earning a doctorate in number theory, he taught at a girl’s school in Berlin. He subsequently earned his habilitation degree in 1869. Cantor proved the rationals $\mathbb{Q}$ are countable in 1873 and that the reals $\mathbb{R}$ are uncountable in 1874. (He also showed that algebraic numbers are countable while the transcendental numbers are uncountable.) While contributing major results to number theory, analysis, and topology, Cantor is considered one the founders of set theory. In 1915, Cantor wrote the seminal book *Contributions to the Founding of the Theory of Transfinite Numbers* (pdf online).

The Cantor Set

Cantor described the *Cantor set* in 1883 as an example of a *perfect set* that is nowhere dense. (The set had been discovered in 1875 by H. J. S. Smith.) Our interest is in comparing the Cantor set to $\mathbb{Q}$. We’ll build the set in stages.

Start with $C_0$ being the closed interval $[0, 1]$. The total length of $C_0$ is 1.

To generate $C_1$, remove the “middle-third;” that is, remove the open interval $(1/3, 2/3)$ from $C_0$ to form $C_1$. So $C_1 = [0, 1/3] \cup [2/3, 1]$. The total length of $C_1$ is 2/3.

Now $C_2$ is made from $C_1$ by removing the middle thirds of each interval. Hence $C_2 = [0, 1/9] \cup [2/9, 1/3] \cup [2/3, 7/9] \cup [8/9, 1]$. The total length of $C_1$ is 4/9. See Figure 1.

![Figure 1: Constructing the Cantor Set](image)

---

1The old German doctorate is between our masters and PhD degrees.

2Roughly equivalent to our PhD.
Continue this procedure to generate the Cantor set \( C_\infty = C \).

Justify the following observations.

1. Each iteration doubles the number of subintervals.
2. An interval endpoint stays in the set.
3. The number of subintervals in \( C_k \) is \( 2^k \).
4. The length of each subinterval in \( C_k \) is \( 1/3^k \).
5. The total length of \( C_k \) goes to \( \infty \) as \( k \) goes to infinity.
6. The length of \( C_{k+1} \) is the length of \( C_k \) minus \( 2^k / 3^{k+1} \).
7. The total of the lengths removed is equal to \( \infty \).
8. The total length of \( C \) is 1 minus the total of the lengths removed and is equal to \( \infty \).

Since we are removing the middles of the intervals, the endpoints remain in the sets, doubling the number of endpoints in each iteration. (Recall Observation 1.) This doubling can be used to create a bijective function from \( C \) onto \([0, 1]\).

9. The Cantor set is uncountable.
10. Since \( \mathbb{Q} \) is countable, then \( C \cap \mathbb{Q} \) is also countable. Therefore, there must be uncountably many irrational numbers in \( C \).

More Observations

- An explicit formula for the Cantor set can be written in terms of the removed middle-thirds.

\[
C = [0, 1] - \bigcup_{m=1}^{\infty} \left[ \bigcup_{k=0}^{3^m-1} \left( \frac{3k+1}{3^m} \right), \frac{3k+2}{3^m} \right]
\]

- The Cantor set can be viewed as a fractal since each stage mirrors the previous (self-similarity).

- Infinity is strange: The Cantor set contains the same number of points as \([0, 1]\), the interval from which it is taken, yet \( C \) contains no interval.

Read about Cantor and other important mathematicians at the “The MacTutor History of Mathematics archive” website at www-gap.dcs.st-and.ac.uk/~history.