Group Project: Fibonacci Numbers & GCDs

Background

Leonardo da Pisa (1170–1250), known as Fibonacci, introduced Hindu-Arabic numerals to Europe. Leonardo had learned Arabic mathematics while living in Algeria as a youth. In his text Liber abaci (The Book of Calculation), he presented a problem:

A certain man put a pair of rabbits in a place surrounded on all sides by a wall. How many pairs of rabbits can be produced from that pair in a year if it is supposed that every month each pair begets a new pair which from the second month on becomes productive?\(^1\)

The answer to the question is the sequence we call Fibonacci Numbers.

<table>
<thead>
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<th>2</th>
<th>3</th>
<th>4</th>
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<tbody>
<tr>
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<td>0</td>
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<td>1</td>
<td>2</td>
<td>3</td>
<td>5</td>
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<tr>
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<td>0</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>Total pairs</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>5</td>
<td>8</td>
</tr>
</tbody>
</table>

Table 1: Fibonacci’s Rabbits

There is an incredible richness to this sequence—it appears all over nature. (An interdisciplinary project for secondary students is researching and reporting on a topic such as Fibonacci and sunflowers, and nautilus shells, &c.) Fibonacci’s sequence has been extensively studied. Look at the *The Fibonacci Quarterly*, the official publication of the The Fibonacci Association to see current work.

The Fibonacci Sequence

Fibonacci began his sequence with 1 (pair of rabbits). Thus

$$1, 1, 2, 3, 5, 8, 13, 21, 33, \ldots$$

Now, we often start with 0, 1. Generalized Fibonacci sequences choose two initial values \(a\) and \(b\), and continue the pattern. Expressing the pattern as a recurrence is very convenient and helps us to analyze and find properties of the sequence.

**Definition 1.** Define \(F_n\) to be the \(n\)th Fibonacci number. Set \(F_1 = 1\) and \(F_2 = 1\). Then

$$F_n = F_{n-1} + F_{n-2} \quad \text{for } n > 2$$

This recurrence formula will figure prominently in nearly all of the proofs you will write below.

We will now turn to proving a sequence of propositions that culminates in a surprising result relating the Fibonacci numbers and greatest common divisors.

\(^1\)The original 1202 text reads: *Quot paria coniculorum in uno anno ex uno pario germinantur. Qvidam posuit unum par cuniculorum in quodam loco, qui erat undique partite circumdatus, ut sciret, quot ex eo paria germinarentur in uno anno: cum natura eorum sit per singulum menem alid par germinare; et in secundo mense ab eorum natiuitate germinant.*
The Project

We will step through several propositions leading to the main result of this project, an unexpected relation concerning GCDs and Fibonacci numbers. Your task is to fill in the proofs carefully and write a narrative connecting the propositions.

**Lemma 1.** Let $a$, $b$, and $c \in \mathbb{N}$. If $c | b$, then $\gcd(a, c) = \gcd(a + b, c)$.

*Proof.* Direct computation. □

**Proposition 2.** For $n \in \mathbb{N}$, it follows that $\gcd(F_n, F_{n+1}) = 1$; i.e., that $F_n$ and $F_{n+1}$ are relatively prime.

*Proof.* The recursion formula, the lemma, and strong induction. □

**Proposition 3.** For $m$ and $n \in \mathbb{N}$, we have $F_{m+n} = F_{m-1}F_n + F_mF_{n+1}$.

*Proof.* Fix a value of $m$ and do induction on $n$. □

**Proposition 4.** For $m$ and $n \in \mathbb{N}$, we have $F_m | F_{mn}$.

*Proof.* Fix a value of $m$ and do induction on $n$ by assuming true for $mk$ and showing true for $m(k+1) = mk+m$ using Proposition 3. □

**Proposition 5.** If $m = qn + r$, then $\gcd(F_m, F_n) = \gcd(F_n, F_r)$.

*Proof.* Apply Propositions 1 to 4 (in some order) to $\gcd(F_m, F_n) = \gcd(F_{qn+r}, F_n)$. □

We have all the pieces ready; now put them together to yield

**Theorem 6.** The $\gcd(F_m, F_n) = F_{\gcd(m,n)}$.

*Proof.* The previous proposition(s) and remember the development of the Euclidean algorithm. □

Project Report

Your project report should contain:

- Your and your partner’s names at the top right of the first page.
- Carefully crafted proofs of the 6 results stated above.
- A narrative connecting the propositions together.

Your project grade will be based on both the correctness of the proofs and your writing.