Asymptotic Notation


Let \( x \) be a variable tending to a limit \( x_0 \) or infinity. Also, let \( \phi(x) \) be a positive function and \( f(x) \) be any function. Then [following Hardy and Wright (1979)] define

1. \( f(x) = O(\phi(x)) \) to mean that \( |f(x)| < A\phi(x) \) for some constant \( A \) and all values of \( x \) near \( x_0 \) or, if \( x_0 = \infty \), all values \( x > M \) for some constant \( M \).

2. \( f(x) = o(\phi(x)) \) to mean that \( \lim f(x)/\phi(x) = 0 \) as \( x \to x_0 \) or \( \infty \).

3. \( f \sim \phi \) to mean that \( \lim f(x)/\phi(x) = 1 \) as \( x \to x_0 \) or \( \infty \).

Note: \( f(x) = o(\phi(x)) \) implies and is stronger than \( f(x) = O(\phi(x)) \).

The term Landau symbols\(^1\) is sometimes used to refer the big-\(O\) and little-\(o\) notations.

REFERENCES:


Examples

1. If \( p(x) = 7x^5 + x^4 - 2x^2 + 4 \), then \( p(x) = O(x^5) \) as \( x \to \infty \).

2. If \( f(x) = \sin(x^2) \), then
   
   a. \( f(x) = O(1) \) as \( x \to \infty \).
   
   b. \( f(x) = O(x^2) \) as \( x \to 0 \).

   c. \( f(x) = o(x) \) as \( x \to 0 \), but \( f(x) \neq o(x^2) \) as \( x \to 0 \).

3. If \( n \in \mathbb{N} \), then
   
   a. \( x^n = O(e^x) \) as \( x \to \infty \).
   
   b. \( x^n = o(e^x) \) as \( x \to \infty \).

   c. \( \ln(x) = o(x^n) \) as \( x \to \infty \).

4. If \( f(x) = O(\phi(x)) \) and \( g(x) = O(\psi(x)) \), then \( f(x) + g(x) = O(\phi(x) + \psi(x)) \).

5. Taylor’s theorem. If \( f \) is \( n \)-times differentiable on \([a,b]\) and \( x, x_0 \in [a,b] \), then for \( h = x - x_0 \)

\[
f(x) = f(x_0 + h) = f(x_0) + f'(x_0) \cdot h + \cdots + \frac{f^{(n)}(x_0)}{n!} \cdot h^n + O(h^{n+1})
\]

\(^1\)The symbol \( O(x) \) first appeared in the second edition of Bachmann’s Analytic Number Theory [Bachmann 1894 (in German)], and Landau obtained this notation from Bachmann’s book (Landau 1909, p. 883). However, the symbol \( o(x) \) did originate with Landau (1909) in place of the earlier notation \( \{x\} \).