Project 14

Summations and Inductive Verification

For this project you will need familiarity with the commands:

<table>
<thead>
<tr>
<th>sum</th>
<th>Sum normal</th>
</tr>
</thead>
<tbody>
<tr>
<td>factor</td>
<td>rhs</td>
</tr>
<tr>
<td>:=</td>
<td>-&gt;</td>
</tr>
</tbody>
</table>

Background

Maple computes “indefinite summations” just as it computes indefinite integrals. Verifying the correctness of an indefinite integral is done, by virtue of the Fundamental Theorem, by simply computing its derivative. Traditionally the validity of an indefinite summation formula is established by mathematical induction\(^1\). Mathematical induction is one of the most fundamental techniques of proof in mathematics. One’s first encounter with induction is usually to verify formulas. As an example:

By observing

\[
\begin{array}{c}
1 + 3 = 4, \\
1 + 3 + 5 = 9, \\
1 + 3 + 5 + 7 = 16, \\
\ldots
\end{array}
\]

one might conjecture the sum of the first \(n\) odd numbers to be the \(n\)th square. Algebraically stated, this is the formula:

\[1 + 3 + 5 + \cdots + (2n - 1) = n^2\]

\(^1\)See, e.g., Chapter 3 of *Calculus of Finite Differences* by Charles Jordan, Chelsea Publishing Company, New York, 1947.
Inductive proofs proceed in two parts. The first part is to show

- **IF** the formula above were true for some integer \( k \),
- **THEN** the corresponding formula for the next integer \((k+1)\) must likewise be true.

The second part is to verify, as we have above for \( n = 2, 3, \) and \( 4 \), that

The formula is valid for some (small) number \( n \).

For the example above, we can complete the proof in the following fashion:

Suppose there were an integer \( k \) for which \( 1 + 3 + 5 + \cdots + (2k-1) = k^2 \), then look to the next larger sum \( 1 + 3 + 5 + \cdots + (2k-1) + (2k+1) \) We can — by our assumption about the formula holding for \( k \) — replace \( 1 + 3 + 5 + \cdots + (2k-1) \) with \( k^2 \), giving us the equation:

\[
1 + 3 + 5 + \cdots + (2k-1) + (2k+1) = k^2 + (2k+1) = (k+1)^2
\]

This completes the proof for all values of \( n \) greater than 2. The validity of the formula for \( n = 1 \) is established by the not very enlightening remark that \( 1 = (1)^2 \). This remark suffices, together with preceding argument, to establish that for all positive integers \( n \)

\[
\sum_{i=1}^{n} (2i - 1) = n^2
\]

One need never have computed the summations for \( n = 2, 3, \) and \( 4 \).

The formal statement of mathematical induction is as follows:

Suppose that \( P(n) \) is a statement about the positive integer \( n \). If it can be shown that

- the validity of the statement \( P(k) \) implies the validity of the statement \( P(k+1) \),

and that

- \( P(1) \) is valid,

then it follows that \( P(n) \) is true of all positive integers.

In this project, you will:

1. Use Maple’s `sum` command to supply a formula.

2. Use Maple to carry out the algebraic computations necessary to establish that the validity of \( P(k) \) implies the validity of \( P(k+1) \).

3. Verify that \( P(1) \) is indeed true.

The following is a record of a Maple session. In this session the “inert” command `Sum` and the `->` are used to construct mathematical equations as propositions of \( n \). The commands `rhs` and `lhs` are used to separate the right and left hand sides of equations. The command `value` is used to evaluate inert sums.
We use the \texttt{Sum} command (capital \texttt{S}) to obtain sigma notation statements:

\begin{equation}
\sum_{i=1}^{3} i^3
\end{equation}

The \texttt{sum} command (small \texttt{s}) gives values and formulas:

\begin{equation}
\text{sum}(i^3, i=1..3)
\end{equation}

Correct! \(1 + 8 + 27 = 36\).

\begin{equation}
s := \text{sum}(i^3, i=1..n)
\end{equation}

\begin{equation}
\frac{1}{4}(n + 1)^4 - \frac{1}{2}(n + 1)^3 + \frac{1}{4}(n + 1)^2
\end{equation}

\begin{equation}
\text{factor}(s)
\end{equation}

\begin{equation}
\frac{1}{4}n^2(n + 1)^2
\end{equation}

It is never clear before hand which form of an expression will be most useful.

Now construct the proposition \(P(n)\) for an inductive proof with:

\begin{equation}
P := n \rightarrow \text{Sum}(i^3, i=1..n) = \text{sum}(i^3, i=1..n)
\end{equation}

\begin{equation}
\sum_{i=1}^{n} i^3 = \sum_{i=1}^{n} i^3
\end{equation}

\begin{equation}
P(3)
\end{equation}

\begin{equation}
\sum_{i=1}^{3} i^3 = 36
\end{equation}

\begin{equation}
eq := P(n)
\end{equation}

\begin{equation}
\sum_{i=1}^{n} i^3 = \frac{1}{4}(n + 1)^4 - \frac{1}{2}(n + 1)^3 + \frac{1}{4}(n + 1)^2
\end{equation}

\begin{equation}
\text{factor}(eq);
\end{equation}

\begin{equation}
\sum_{i=1}^{n} i^3 = \frac{1}{4}n^2(n + 1)^2
\end{equation}

Now we proceed with an inductive proof of \(P(n)\).
Suppose that for some positive integer \( k \), \( P(k) \) is true. Specifically assume the truth — for one particular \( k \) — of:

\[
\sum_{i=1}^{k} i^3 = \frac{1}{4}(k + 1)^4 - \frac{1}{2}(k + 1)^3 + \frac{1}{4}(k + 1)^2
\]

Adding \((k + 1)^3\) to both sides preserves equality so:

\[
\sum_{i=1}^{k} i^3 + (k + 1)^3 = \frac{1}{4}(k + 1)^4 + \frac{1}{2}(k + 1)^3 + \frac{1}{4}(k + 1)^2
\]

By inspection, the left hand side of the above equation is the sum \(1^3 + 2^3 + \cdots + (k + 1)^3\). If the right hand side of the above can be shown to be equal to \( \text{rhs}(P(k+1)) \), we have established that the validity of \( P(k) \) implies the validity of \( P(k+1) \).

Select the right hand side of our last equation for comparison to \( \text{rhs}(P(k+1)) \):

\[
\frac{1}{4}(k + 2)^2 (k + 1)^2
\]

and compare to the factored form of \( \text{rhs}(P(k+1)) \):

\[
\frac{1}{4}(k + 2)^4 - \frac{1}{2}(k + 2)^3 + \frac{1}{4}(k + 2)^2
\]

They match! To complete the proof we need only determine if \( P(1) \) is true.

\[
\sum_{i=1}^{1} i^3 = 1
\]

And now, for the truly lazy:

\[
\text{value}(P(1)) = 1 = 1
\]
Project Report

1. Use Maple to produce and verify closed forms for the indefinite summations:
   \[ \sum i^4, \quad \sum i^5, \quad \sum \frac{1}{i(i+1)}, \quad \sum \frac{1}{i(i+1)(i+2)}, \quad \sum x^i \]

2. Comment on the following argument.

   Define the proposition \( P(n) \) by
   \[ P := n \to 1 + 2 + 3 + \cdots + n = \frac{n^2 + n + 1}{2} \]

   Proof:
   Suppose for some integer \( k \): \( 1 + 2 + 3 + \cdots + k = \frac{(k^2 + k + 1)}{2} \),
   then look to the next larger sum \( 1 + 2 + 3 + \cdots + k + (k + 1) \) We can — by our assumption about \( P(k) \) — replace \( 1 + 2 + 3 + \cdots + k \) with \( \frac{(k^2 + k + 1)}{2} \) giving us the equation:
   \[ 1 + 2 + 3 + \cdots + k + (k + 1) = \frac{k^2 + k + 1}{2} + (k + 1) \]
   \[ = \frac{k^2 + k + 1}{2} + \frac{2k + 2}{2} \]
   \[ = \frac{k^2 + 3k + 3}{2} \]
   \[ = \frac{(k^2 + 2k + 1) + (k + 1) + 1}{2} \]
   \[ = \frac{(k + 1)^2 + (k + 1) + 1}{2} \]

   and, hence, \( P(k + 1) \) holds. This completes the proposition that the validity of the statement \( P(k) \) implies the validity of the statement \( P(k + 1) \).
   Thus \( P(n) \) is valid for all \( n \).

Extension

The trigonometric identity
\[ \frac{1}{2} + \sum_{j=1}^{n} \cos(jx) = \frac{\sin \left( (n + \frac{1}{2})x \right)}{2 \sin \left( \frac{x}{2} \right)} \]

is fundamental to the study of Fourier series.
Prove this identity with induction by

1. Define $P(n)$.

2. Prove the validity of $P(k+1)$ follows from assuming $P(k)$ is true by considering:

   (a) $\text{rhs}(P(k+1)) = \text{rhs}(P(k)) + \cos((k+1) \ast x)$

   Why is this the correct expression to study?

   (b) Multiply both sides by the denominator to clear fractions.

   (c) Now use \texttt{combine(..., trig)} to reduce the expression using the trigonometric sum and product formulas.

3. Establish that $P(0)$ is true.

   Is it acceptable to use $P(0)$ rather than $P(1)$?

**Report Requirements**

A minimal project report will include:

- Maple output with supporting argumentation (this can, of course, be supplied within Maple) that inductively verifies closed form formulas for the five sums listed in Problem 1 above.

- A well-written English language critique of the argument in Problem 2 that seeks to establish a formula for the sum of the first $n$ positive whole numbers.