Project 13

The Catenary: An Application

For this project you will need familiarity with the commands:

<table>
<thead>
<tr>
<th>plot</th>
<th>int</th>
<th>diff</th>
</tr>
</thead>
<tbody>
<tr>
<td>subs</td>
<td>simplify</td>
<td>solve</td>
</tr>
<tr>
<td>fsolve</td>
<td>?</td>
<td></td>
</tr>
</tbody>
</table>

You may also need to use the *search interval limiting* capability of *fsolve* and the *metric* option of *convert*.

Background

In 1691, Christiaan Huygens and Gottfried Leibniz correctly settled a question raised by Galileo Galilei

> What is the form a perfectly flexible, inextensible chain will assume when suspended by its ends and acted upon by gravity alone?

(Galileo had incorrectly thought it was a parabola. Parabolas are the shape assumed by a chain bearing a uniformly distributed load such as the cables suspending the Oakland Bay Bridge.) Huygens and Leibniz reduced the question to finding the solution of the differential equation:

\[
a \frac{d^2y}{dx^2} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}
\]

where \(y(x)\) is the height above the ground\(^1\). Their solution is known as the catenary from the Latin *catenarius* meaning “relating to a chain.” In 1695, the curve arises again as the description given by the Bernoulli brothers\(^2\) of the shape (cross section) of a wind-filled rectangular sail.

\(^1\)The constant \(a\) is determined by \(a = \frac{h}{w}\) where \(w\) is the weight per unit length, and \(h\) is the horizontal tension.

\(^2\)In the 17th and 18th century the Bernoulli family produced a remarkable string of first rate mathematicians and physicists who collectively left the family name all over physics and analysis.
Today, we denote the solution of the differential equation as the hyperbolic cosine. To see

\[ y = a \cosh(x/a) \]

is a solution of the differential equation, enter:

```plaintext
> diffeq := a*(diff(F(x),x,x)) = sqrt(1+(diff(F(x),x))²)

then

> deF := subs(F(x)=a*cosh(x/a), diffeq)

and

> simplify(deF, symbolic)

(The `symbolic` option of `simplify` assumes the arguments of square roots are positive. Try the statement without the option.)

Graph the hyperbolic cosine with

```plaintext
> plot(cosh(x), x=-2..2, y=0..4)
```

To see how Galileo’s intuition had misled him, examine the images:

```plaintext
> plot({cosh(x), x²+1}, x=-3..3)
> plot({cosh(x), x²+1}, x=-8..8, y=0..72)
```

What is

\[ \lim_{x \to \infty} \frac{\cosh(x)}{x²+1} \]

The image of \( y = \cosh(x) \) is the curve of a telephone wire.

**Project Report**

The Otsego Electric Company has assigned your work group the following task.

A broad-band communication link must be established between the Administration Building and Building G. Determine the number of poles and the length of the wire needed to connect the two buildings at the Gaylord Assembly Plant 3 km apart, given that the distance between successive utility poles will be 100 yd. The poles support the wire 55 ft above the ground with the center sag being no more than 15 ft. Cable will cost $0.75 per foot. Poles will cost $45 each. Determine the total cost of materials.
We solve this problem by reducing it to manageable parts. First, we determine an equation that models the wire hanging between two poles; second, we calculate the length of wire between two poles. Finally, we determine the cost of materials for the entire project.

I. Equation of the Wire

To develop the equation of the curve, we set the origin at the center where the dip of the wire is the greatest.

Abstracting from the physical situation yields:

\[ y = a \cosh\left(\frac{x}{a}\right) \]

First, the equation for the curve of the wire must be found:

\[ f := x \rightarrow a \cosh\left(\frac{x}{a}\right) + b. \]

At the beginning of any computation, one should obtain reasonable estimates for comparison with the final answer. Since the curve is inside the triangle we can obtain a lower estimate for the length of the wire by calculating the length of the hypotenuse \( \sqrt{15^2 + 150^2} \) and an upper estimate from the sum of the two legs, \( 15 + 150 \); that is,

\[ 150.7 \leq \frac{1}{2} (\text{length of wire between two poles}) \leq 165 \]

Define \( f(x) \) with the Maple statement

\[ f := x \rightarrow a \cosh\left(\frac{x}{a}\right) + b; \]

Use fsolve, possibly with a range restriction, to determine \( a \) and \( b \) that make \( f(0) = 40 \) and \( f(150) = 55 \).
1. Determine the function \( f(x) \) modeling the cable.

**Question 1** Check \( f(0) \) and \( f(150) \). Is \( f(x) \) giving the correct height for the wire?

II. Length of Wire

Our next step will be to compute the arclength using the formula

\[
L(u) = \int_{0}^{u} \sqrt{1 + (f'(x))^2} \, dx
\]

To that end, define

\[
Df := \text{D}(f) \\
ds := \text{sqrt}(1 + (Df(x))^2)
\]

It is good practice to simplify an integrand before integrating.

\[
ds := \text{simplify}(ds, \text{symbolic})
\]

And now to find the cable’s length, enter

\[
L := \text{int}(ds, x=-150..150)
\]

2. Find the length of the cable.

**Question 2** Is the computed length in line with our estimate?

\[
150.7 \leq \frac{1}{2} (\text{length of wire between two poles}) \leq 165
\]

III. The Number of Poles

The last step is to determine the number of poles necessary to span three km and calculate the total length of wire needed.

3. Find the total length of wire and the number of utility poles necessary to string the cable between the buildings.

Unless you remember your conversions from grade school, you will find the Maple command `convert(..., \text{metric})` useful.

IV. Total Cost

Finally, assemble all the information produced to put the various costs together.

4. Determine the total cost of materials for the project.
Extension

1. The differential equation modeling a cable carrying a uniform load of $k$ pounds per horizontal foot is given by

$$a \frac{d^2y}{dx^2} = k$$

Show this differential equation has a one-parameter family of parabolas as its solution when assuming $y(0) = 0$ and $y'(0) = 0$.

2. The Pursuit differential equation is closely related to the catenary equation. If the target has velocity $v_0$ and the pursuer has velocity $v_1$, then the differential equation becomes

$$x \frac{d^2y}{dx^2} = \frac{v_0}{v_1} \sqrt{1 + \left( \frac{dy}{dx} \right)^2}.$$ 

Define this differential equation in Maple with $v_0/v_1 = 1/2$ and use `dsolve` with the initial conditions $y(1) = 0$ and $y'(1) = 1$ to obtain a solution. After viewing several plots, discuss the change in the solution when the pursuer doubles or triples speed ($v_1$).

Report Requirements

Address your project report to your supervisor Leslie Fusaro. A minimal project report will include:

- English responses to Questions 1 and 2.
- Answers with justifying narratives for Problems 1 through 4.