Mapping Surfaces onto the Unit Sphere
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A brief overview of the Gauss map and mappings to and from the Riemann sphere, and qualitative discussion on the methodology and utility of mappings between surfaces and the unit sphere.

I. Introduction

In 1822, a famous mathematician by the name of Gauss (apparently Gauß properly in German) was presented an award for, among other things, the idea of mapping one surface onto another in a manner such that their local intrinsic behavior is unchanged. This marked the beginning of formalized, systematic, useful, and mathematically elegant mappings from surface to surface.

The most direct and useful application of mapping at the time was the practical problem of mapping a curved Earth onto flat sheets of paper. However, more abstract uses are available. If a mapping preserves critical qualities of the original surface, a highly complex or unfamiliar surface can be treated as a more familiar surface whose properties are readily known.

A simple example of a mapping is “unrolling” a cylinder to deal with it as a flat surface. In this paper, we will discuss various methods of mapping surfaces to the unit sphere. While this direction of mapping is not particularly interesting to cartographers, geometers and physicists find the unit sphere a mathematically familiar environment of comfortably finite dimension, and less convenient infinite surfaces may be mapped to it in several well defined ways, depending on what properties need to be preserved.

II. The Gauss map

In 1827, Gauss published a work titled *Disquisitiones generales circa superficies curvas*, which translates as “Investigations of the general behavior of curved surfaces.” In this work, he described what is now known as the Gauss map.

The Gauss map is closely tied with other basic differential geometry features of a curve, such as the Gaussian curvature and shape operator; it was not a random addition to Gauss’s work.

The simplest conceptual approach to the Gauss map is to treat it as the set of unit normal vectors of the surface in question. Each point \((x,y,z)\) on the original surface maps to the point \((a,b,c)\), where the normal vector is \(\mathbf{n}=(a,b,c)\).

As the normal vector \(\mathbf{n}(x,y,z)\) is not necessarily unique, the Gauss map is not necessarily one-to-one; as there is not, for every curve, a point \((x,y,z)\) for each \((a,b,c)\) such that \(\mathbf{n}(x,y,z)=(a,b,c)\), the Gauss

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1. Such practical mappings were systematized in primitive empirical form by Mercator and other early cartographers; however, the generalized geometry for dealing with mappings in an elegant analytic form were not developed until the 19th century.
2. The term “Gauss map” refers not simply to the function (i.e., the map from the surface to the unit sphere), but also the resulting image on the unit sphere, which may or may not occupy the entire sphere.
map is not necessarily onto. Gauss maps are, in fact, generally a fraction of the unit sphere, frequently with double or triple overlaps and even outright singularities where an infinite number of points map to a single point on the unit sphere. To briefly describe several familiar curves:

- The flat plane. Every point on a flat plane is mapped to the same point on the unit sphere using a Gauss map.
- A cone of angle \( \theta < 2\pi \) with an axis \( \mathbf{a} \) maps to the circle of points at angle \((\pi - \theta)/2\) around the same axis \( \mathbf{a} \) within the unit circle.
- A simple rotated paraboloid, represented by \( z = ar^2 + c \) in cylindrical coordinates, maps to either the upper or lower hemisphere of the unit sphere, depending on the sign of \( a \). This mapping has, we will note, no relationship to \( c \), and the only relevance of \( a \) is in giving the change in density on the Gauss map.
- A sphere is simply compressed or expanded to the radius of 1 and centered at the origin.
- The Gauss map of a catenoid is simply a sphere, mapping one to one.
- The Gauss map of a normally oriented torus overlaps the unit sphere twice – with the addition of an infinite number of points mapping to \((0,0,1)\) and \((0,0,-1)\). These points are the top and bottom “circles.”
- If the same torus is oriented about the \( x \) axis instead of the \( z \) axis, the points \((1,0,0)\) and \((-1,0,0)\) become singularities rather than \((0,0,1)\) and \((0,0,-1)\).

Notice anything particular? Translation of the surface does not affect the appearance of its Gauss map; expansion of the surface only changes the magnitude of the derivative of the Gauss mapping function itself, and not its general behavior on the curve. The image of the map is completely unaltered by scaling or translating the curve; re-orienting the surface in a different direction simply rotates the Gauss map.

In other words, the Gauss map itself is an intrinsic beast, its shape dependent solely on the nature of the curve. Know a function's Gauss map, and you know how the surface behaves.

The mapping function itself \( G \) has a derivative \( G' \) equal to the -\( S \), where \( S \) is the shape operator of the curve, and the determinant of the mapping function at some point \((x,y,z)\) is equal to the Gaussian curvature at \((x,y,z)\). Similarly, mean curvature at \((x,y,z)\) on the curve is equal to the negative of one half the trace of \( G(x,y,z) \).

**III. The Riemann sphere**

Bernhard Riemann was born the same year that Gauss was awarded for coming up with the idea of a mapping preserving local behavior from one surface to another. His contributions to differential geometry were substantial.

A Riemann surface is a well behaved complex manifold. Spheres, toruses, paraboloids, planes, and most other surfaces of interest can be defined as Riemann surfaces using the appropriate parameterization. What is interesting about Riemann surfaces is that there exist holomorphic maps

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3 The shape operator \( S \) is a 2x2 matrix whose terms are two vectors representing the covariant derivatives of our local variables, \(-\nabla_UV, -\nabla_VU\), i.e., an assortment of terms of a combination of \( E, F, G \) and \( e, f, g \). The negative sign in the original \( S \) term produces the negative sign in the derivative of \( G \).
from each Riemann surface to another Riemann surface; that is to say, there exists a function \( F \) such that for any analytic curve\(^4\) on a surface \( S_1 \), that same curve on \( S_2 = F(S_1) \) behaves exactly the same on \( S_2 \) as it did \( S_1 \).

The Riemann sphere is not necessarily a unit sphere by definition; however, it is most useful to consider the Riemann sphere as a unit sphere. The Riemann sphere contains a single point representing infinity and a completely uncompressed point at the local origin opposite infinity. To the right, we see the mapping from a line to a circle of radius \( R \); to extend this to a mapping from a plane onto a unit sphere, consider \( R=1 \) and rotate the figure about the axis.

The Riemann sphere is useful as a highly compact map of an infinite surface. Mapping another Riemann surface to the Riemann sphere is one­to­one and onto – even for an infinite sphere.

Curves are of course distorted visually, but the new representation on the sphere is useful. An infinite line through the origin of the \( xy \) plane becomes a finite line of longitude mapped on the Riemann sphere via the above method, for example, while a circle centered at the origin becomes a line of latitude on the sphere (\( x^2 + y^2 = 4 \) is the equator for a unit sphere.)

\[ \text{Illustration 1: Mapping an infinite line onto a circle of radius } R, \text{ from } \]
\[ \text{http://en.wikipedia.org/wiki/Riemann_sphere} \]

\[ x' = \frac{-2Rx}{2R - y} \]

\[ (x, y) \]

\[ \text{center of projection} \]

\[ \text{point of tangency} \]

IV. Requirements for mapping onto the unit sphere

A surface may be mapped in some useful fashion to the unit sphere \textit{if and only if} the surface is \textit{orientable}. Riemann surfaces must be orientable in order to be Riemann surfaces; Gauss maps are defined only for orientable surfaces\(^5\). This requirement is not unique to Gauss maps and maps to the Riemann sphere.

If a surface is orientable, then a figure inscribed on the surface may be translated along local coordinates \( u \) and \( v \) without the translations ever becoming equivalent to a reflection about an axis of the figure. Orientable surfaces include spheres, toruses, paraboloids, mountain ranges, etc; non-orientable surfaces include Möbius strips, Klein bottles, and other “one-sided” figures, which tend to be difficult to deal with in a similar fashion to more familiar surfaces.

An orientable surface may be defined (per the above) as a surface such that there exists a continuous mapping \( f \) from the unit sphere to the surface\(^6\). From this we know that any map of a surface

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\( ^4 \) An analytic curve is a well-behaved curve – essentially, differentiable and continuous throughout. Polynomials, exponentials, trig functions restricted to appropriate domains, etc, are all analytic.

\( ^5 \) It would be possible to take the unit normal at each point locally for these surfaces, in many cases. Eventually, however, you find yourself at the same point and mapping an opposite unit normal, meaning that \( G \) is simply no longer a function as defined in traditional mathematics.

\( ^6 \) This may seem somewhat circular. However, in formal terms, this turns out to be one of the most useful ways of testing orientability in algebraic terms.
S onto the unit sphere will be non-continuous if S is a non-orientable surface – and if the map is non-continuous, it will not preserve useful properties of the surface or curves within the surface.

V. Utility of mapping onto the unit sphere

It is useful to map onto the unit sphere if and only if both of two conditions both hold.

First, the original surface must be in some fashion poorly understood or difficult to manipulate. A mapping may be used to investigate the properties of the surface itself, deal with behavior of objects or functions intrinsic to the surface, etc. There is usually little point in mapping the complex plane to the Riemann sphere unless you wish to deal with properties relating to infinite functions, which are a pain to deal with conventionally.

In general, a map from an infinite surface to the unit sphere is most useful in how it reduces an infinite surface into a finite domain, allowing a mathematician to easily visualize or graph exactly what s/he is dealing with. The unit sphere is a very well understood surface, and the mathematics of dealing with the unit sphere reduce remarkably quickly to elementary trigonometry.

Second, the mapping itself must preserve or present useful properties. The mapping is of no utility if it does not in some fashion preserve the local behavior of the surface; if the mapping is not locally invertible, then the any manipulations performed after the mapping cannot be re-applied to the original surface. A good example of this is graphing analytic functions on a Riemann sphere, or calculating Gaussian curvature from the mapping function G.

An example of a geometrically useless mapping would be a map which takes each point \((u,v)\) on the original surface \(S\) and maps it to an apparently random point \((\phi,\theta)\) on the sphere non-continuously. This mapping would be equivalent to an encryption of the real numbers – useful for hiding information, but useless for learning about the properties of the original surface or uncovering new information about curves within the surface.

VI. Bibliography and further reading


Biographical information on Gauss from an article by J. J. O'Connor and E. F. Robertson, available at http://www-groups.dcs.st-and.ac.uk/~history/Printonly/Gauss.html

Biographical information on Riemann from an article by Dr. Dörte Haftendorn, available at: http://www.fh-lueneburg.de/u1/gym03/englpage/chronik/riemann/riemann.htm and http://www.fh-lueneburg.de/u1/gym03/englpage/chronik/riemann/jugend/jugend.htm

For a particularly fine graphical representations of the Gauss map and mapping function for a particular saddle function, see http://www.math.union.edu/~dpvc/TFB/ICMS-poster/cgm/welcome.html

For discussion of orientable surfaces, please see Wikipedia’s excellent Orientable article at http://en.wikipedia.org/wiki/Orientable