Gottfried Wilhelm Leibniz made many contributions to modern day Calculus, but none of his contributions may have been as important to his development of his Calculus than his work with the Harmonic triangle. This work, and his subsequent work with series, partial sums, and sums of differences may have driven Leibniz towards his ultimate definition of the Calculus. We will delve into the harmonic triangle shortly, but first, a brief history of Leibniz leading up to this point.

Leibniz was born in Leipzig, Germany, in 1646. His father was a professor of Moral Philosophy at the University of Leipzig, but died when G.W. Leibniz was only six years old. After his father's death, Leibniz began reading from his father's extensive library of books about all sorts of subjects. At the age of fifteen, Leibniz became a law student at the same school his father had been a professor, the University of Leipzig. He obtained his bachelors degree at the age of seventeen, and his doctorate in Law from the University of Altdorf, Nuremburg, at the age of twenty. Leibniz taught himself mathematics, by reading papers and journals, and meeting many mathematicians in his early career as counsel and legal advisor to Kings and Princes. In 1672, Leibniz visited Paris as a diplomat by invitation of the French foreign minister. Leibniz remained in Paris for much of the next four years.

In the fall of 1672, Leibniz met and became friends with Christiaan Huygens, a Dutch physicist. Huygens became Leibniz's mathematical mentor. It was during this time that Leibniz developed and used the Harmonic Triangle. Originally prompted by a
mathematical challenge from Huygens in 1672, Leibniz's work with the Harmonic Triangle began. Huygens challenged Leibniz to calculate the infinite sum of the reciprocals of the triangular numbers, also known as Pascal's Triangle. Pascal's Triangle as displayed below in (Figure 1), worked with special numbers, in triangular arrangements that had specific properties.

The numbers were of the form: 1, (1+2), (1+2+3), ...
with the general form: \( \frac{i(i+1)}{2} \) for \( i=1,2,... \)

In Pascal's Triangle, each number on the outside is 1, and all other numbers are the sums of the 2 numbers directly above.

\[ \text{Pascal's Triangle} \]

\[
\begin{array}{cccccccccccc}
& & & & & 1 & & & & & & \\
& & & & 1 & & 1 & & & & & \\
& & & 1 & & 2 & & 1 & & & & \\
& & 1 & & 3 & & 3 & & 1 & & \\
& 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1 \\
& 1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 \\
& 1 & & 7 & & 21 & & 35 & & 35 & & 21 & & 7 & & 1 \\
& & 1 & & 8 & & 28 & & 56 & & 70 & & 56 & & 28 & & 8 & & 1 \\
\end{array}
\]

\text{Figure 1.}
Leibniz's Harmonic Triangle

\[
\begin{array}{cccccccc}
 & & & & & 1 & & \\
 & & & & 1 & & 1 & \\
 & & & 1/2 & & 1/2 & & \\
 & & 1/3 & & 1/6 & & 1/3 & \\
 & 1/4 & & 1/12 & & 1/12 & & 1/4 \\
 & 1/5 & & 1/20 & & 1/30 & & 1/5 \\
 & 1/6 & & 1/30 & & 1/60 & & 1/30 & & 1/6 \\
 & 1/7 & & 1/42 & & 1/105 & & 1/140 & & 1/105 & & 1/42 & & 1/7 \\
\end{array}
\]

Figure 2

Huygens challenge, which had been accomplished before, was for Leibniz to calculate the infinite sum of their reciprocals:

\[
\frac{1}{1} + \frac{1}{3} + \frac{1}{6} + \frac{1}{10} + \ldots + \frac{2}{n(n+1)} + \ldots
\]

Leibniz attacked the problem by recognizing that each term:

\[
\frac{2}{i(i+1)} = \frac{2}{i(i-2)}
\]

So the sum could be rewritten as:

\[
(\frac{2}{1} - \frac{2}{2}) + (\frac{2}{2} - \frac{2}{3}) + (\frac{2}{3} - \frac{2}{4}) + (\frac{2}{4} - \frac{2}{5}) + \ldots + (\frac{2}{n} - \frac{2}{n+1}) + \ldots
\]
Simplifying and cancelling, the partial sum of the first \( n \) terms has as it's sum:

\[
2 - \left( \frac{2}{n} + 1 \right) = 2
\]

for \( n = 1 \) to \( \infty \) or in more modern day notation:

\[
\sum_{n=1}^{\infty} 2 - \left( \frac{2}{n} + 1 \right) = 2
\]

Leibniz used his work with the harmonic series and applied it to several other series and generalized it to be the harmonic triangle (Figure 2). On a trip to London in early 1673, Leibniz, proud of his work, discussed his findings with mathematician, John Pell. Leibniz had a general method of dealing with series by constructing the series of differences, especially the sum of the reciprocals of the triangular numbers. Pell quickly pointed out that this had been accomplished before and Leibniz was embarrassed by his ignorance of mathematics of the time. Not to be embarrassed again, he then began a period of serious study of mathematics.

Leibniz quickly realized from the results that forming sums and taking differences are inverse processes of each other. Using these two ideas, as well as, his concentrated study of Pascal's Triangle, his own Harmonic Triangle, and sums and differences of series, shortly led Leibniz to develop the ideas behind what we now call the Fundamental Theorem of Calculus.

Both Pascal's Triangle and Leibniz's Harmonic Triangle have many similar properties. Although there are many more, below are some of the properties that hold for each triangle and their unique relationship with each other.

**Property 1:**
- **Pascal's Triangle** – each entry (not in the first row or column) is the sum of the two terms directly above it.
- **Leibniz's Triangle** – each entry is the sum of the two terms directly below it.
Property 2:
Pascal's Triangle – each entry is the difference of the two terms directly below it.
Leibniz's Triangle – each entry (not in the first row) is the difference of the two terms directly above it.

Property 3:
Pascal's Triangle – each entry (not in the first row or column) is the sum of the number (above and to the right) and all left diagonal terms above.
Leibniz's Triangle – each entry is the sum of the number (below and to the right) and all left diagonal terms below.

Property 4:
Each entry in the nth row of Leibniz's Triangle is the reciprocal of the number generated by multiplying n by the corresponding element in the corresponding nth row of Pascal's Triangle.

Generating both Pascal's triangle and the Harmonic triangle is an exercise with sets of entries involving binomial coefficients.

For Pascal's triangle, we denote the rth entry in the nth row by the binomial coefficient:

\[
\binom{n-1}{r-1} = \frac{(n-1)!}{(r-1)!(n-r)!}
\]

and you add two consecutive entries to give the entry between them in the row below, given by the rule:

\[
\binom{n-1}{r-1} + \binom{n-1}{r} = \binom{n}{r}
\]
For the Harmonic triangle, we denote the $r$th entry in the $n$th row by the binomial coefficient:

\[
\frac{(r-1)!}{n(n-1)(n-2)\ldots(n-r+1)} = \frac{(r-1)!}{n!} \frac{(n-r)!}{n \binom{n-1}{r-1}} = \frac{1}{n \binom{n-1}{r-1}}
\]

and you add two consecutive entries to give the entry between them in the row above, given by the rule:

\[
\binom{n}{r} + \binom{n}{r+1} = \binom{n-1}{r} \quad \text{or equivalently} \quad \binom{n}{r+1} = \binom{n-1}{r} - \binom{n}{r}
\]

Although the Harmonic Triangle was not the most important work of Leibniz, it certainly had a direct impact on his initial ideas about his Calculus. It is also not nearly as well known or used in current classrooms, but could have a definite place in the High School Mathematics classroom, especially when working with rational expressions.
Works Cited


