Homework 3: Curves Continued

You may work alone or in a group of up to 3 people and turn in one per group. Each group must write up their work in their own words and give any credit to others where it is due.

4140 students complete 4 of the first 5 problems. Graduate students complete all 6.

The purpose of homework is to learn and practice computational strategies, concepts, and develop critical thinking and problem-solving skills, so you should try problems on your own. Feel free to talk to me or each other if you are stuck on this assignment, but be sure to acknowledge any sources including each other, except your group members or me. If you know how to do a problem and are asked for help, try to give hints rather than the solution.

Maple Applets (both applets are on the calendar webpage)

spacecurve.mw calculates the Velocity, Acceleration, Jerk, Speed, ArcLength, Curvature, and Torsion
TNBapplet.mw animates the Frenet Frame

Direct Maple Commands:

with(plots):with(VectorCalculus): [needed at the start of any Maple document]

For Question 1:
curve:=plot(x^2, x = -2..2):
osculatingcircle:=implicitplot((x-x0)^2 + (y-y0)^2 - r0^2, x=-1/2..1/2, y=0..1): [fill in constants x0, y0, r0]
display(curve,osculatingcircle); [plots on the same graph]

For Question 2:

helix:=<r*cos(t), r*sin(t), h*t>; [defines a curve. Sub in your specific curve, and rename if desired.]

TNBFrame(helix,t) assuming t::real;
simplify(Curvature(helix,t));
simplify(Torsion(helix,t),trig);

For Question 4:

helix:=<r*cos(t), r*sin(t), h*t>; [defines a curve. Sub in your specific curve, and rename if desired.]
simplify(Curvature(helix,t), trig);
evalf(whatever); [decimal approximation]

ArcLength(helix, t = -10 .. 10);

Cylinder := implicitplot3d(x^2 + y^2 = 1, x = -1 .. 1, y = -1 .. 1, z = 0 .. 10):

curve1:= spacecurve([5*cos(t), 5*sin(t), 3*t, t = 0 .. 7]); [defines the curve]
display(Cylinder, curve1); [plots on the same graph]

General Maple Commands:

Pi; exp(t); cos(t); sin(t);

1. Osculating Circle

a) By hand: Find the radius and equation of the circle that best fits the curve y = x^2 at x_0 = 0.

Hint: You could either parametrize by t and then calculate \( \kappa \) by computing \( \left| \frac{dT}{ds} \right| \) evaluated at \( t = 0 \), OR use the formula from Calc III for the (scalar) curvature of a function \( y=f(x) \), i.e.

\[
\kappa = \frac{f''(x_0)}{(1+f'(x_0)^2)^{\frac{3}{2}}}
\]

After calculating \( \kappa \), find the radius and center of the circle.

b) Plot the curve and the circle in Maple on the same graph (see above for similar commands) and print out your graph and commands.

2. By-hand and Maple Computations

a) Do exercise 1.4.7 on page 31, BOTH by-hand and on Maple. Keep everything with respect to \( t \), and compare the answers, resolving any differences.

Hints: Recall the formulas

\[
\frac{ds}{dt} = |a'(t)|
\]
\[ T(t) = \frac{\alpha'(t)}{|\alpha'(t)|} = \alpha'(t) \]
\[ \kappa = \frac{T'(t)}{dT/ds} \text{ because of the chain rule } \kappa = \frac{dT}{ds} \frac{dt}{ds} = \frac{dT}{dt} \frac{dt}{ds} = \frac{T'(t)}{|\alpha'(t)|} \]

The curvature \( \kappa \) is the length of \( \kappa = |\kappa| \).
\[ N(t) = \frac{\kappa}{|\kappa|} \]
\[ B(t) = T \times N \]

\( \tau \): To calculate the torsion by-hand in this context, where we are not parametrized by arc length, note that
\[ \frac{B'(t)}{|\alpha'(t)|} = \frac{B'(t)}{dt} = \frac{dB}{dt} \frac{dt}{ds} = B'(s) \text{ (using chain rule again), and this equals } -\tau N \]

So compute \( \frac{B'(t)}{|\alpha'(t)|} \) and compare it to \( N \) to find \(-\tau\) and then \( \tau \).

b) Some people define \( \tau \) by \( B'(s) = -\tau N \) and others by \( B'(s) = +\tau N \). Does Maple use plus or minus to define the torsion?

3. Frenet-Equations

If a rigid body moves along the curve \( \alpha(s) \), then the motion of the body consists of translation along \( \alpha \) and rotation about \( \alpha \). The rotation is determined by an angular velocity vector \( \omega(s) \), called the Darboux vector, named for Jean-Gaston Darboux. It satisfies the following:
\[ T'(s) = \omega(s) \times T(s), \quad N'(s) = \omega(s) \times N(s), \quad B'(s) = \omega(s) \times B(s) \]

a) Do exercise 1.3.12 on page 21.

b) Write a short paragraph which interprets each term of \( \omega = \tau T + \kappa B \) in the example of a roller coaster moving along \( \alpha \). (Think of the coaster car and the people inside as the rigid body. A good first example to consider is along a theoretical ferris wheel (circular) hump that stays in that ferris wheel plane (for a bit), where the car is on that part of the track and the front of the car is pointed in the direction of the tangent vector, like approximately in the picture below. Then consider the general case where torsion also comes in.)

Include at least one rough sketch or a picture related to your explanation.

4. Strake

a) Given the setup in the picture above, compute the ideal value for the inner radius of the annulus (your choice of method and your choice of by-hand or Maple to print).
b) Create a model of the annulus and the cylinder, approximately to scale. The model might be a real-life model or a printed Maple plot, for example.

c) A strake is not planar, but annular pieces of flat steel can sometimes be bent without stretching/stressing it too much to produce the strake. Other times a flat annulus could not be used. One can compare the local intrinsic geometry of the strake to the local geometry of the planar annulus as follows:

c1) Compute and show work (your choice of Maple commands to print and/or by-hand work, whatever you prefer here) for the curvatures and arc lengths of the inner and outer edges of the annulus and the corresponding inner and outer edges of the helical strake. So you will have 6 Maple or by-hand computations (inner annulus curvature = inner helix curvature, inner annulus arc length = inner helix curvature, outer annulus curvature, outer annulus arc length, outer helix curvature, outer helix arc length).
   Hint: To calculate the length of the outer annulus, we can set up a proportion: the length of the inner annulus/its radius is the length of the outer annulus/its radius. The curvature of the outer annulus is 1 over its radius.

c2) Do the outer annulus and outer helix computations agree?

c3) What happens if we make the strake very wide compared to the diameter of the cylinder, such as in an auger below. Do you think this be made physically from a flat annulus? Explain why or why not.

5. **Mystery Curve**

   a) First use Maple to help you do (only) part 4 of exercise 1.3.11 on page 21. You can use the same commands as in Question 2, and you will print your work. Be sure to show why they are the same.

   b) For what s values is the curvature defined? For these values, is the curvature positive, negative, zero, or a mixture?

   c) Then continue with exercise 1.5.4 on page 36. See the hint in the back of the book on p. 443 and use this to look for meaning as to what kind of curve this is.

6. **Graduate Problem:** Curve Proof

   Let \( \alpha \) be a curve with \( \kappa > 0 \). Show that if \( \alpha \)'s osculating planes have a point in common, call it \( p \) (i.e. each plane passes through \( p \)), then \( \alpha \) is planar.