Quantifying the Curvature of Curves: An Intuitive Introduction to Differential Geometry

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Abstract

In this paper we introduce the reader to a foundational topic of differential geometry: the curvature of a curve. To make this topic engaging to a wide audience of readers, we develop this intuitive introduction employing only basic geometry without calculus and derivatives. It is hoped that this introduction will encourage many more both to consider this mathematical notion and to develop enthusiasm for mathematical studies.

1 Introduction

Ride a bike or drive a car. Hopefully, well before you end up in a ditch, you will recognize that not all curves in a road are constructed equally: some curves are simply sharper or more curved than others.

The subject of Differential Geometry leads to the measuring (or quantifying) of curve curvature. Unfortunately, formally investigating Differential Geometry at an introductory level requires at least Differential and Integral Calculus and Linear Algebra. Further, Differential Equations, Tensor Algebra, and Calculus as well as Manifold Theory are necessary for more advanced treatments of the subject. However, an intuitive understanding of curve curvature is approachable by high school students and teachers. The main goal of this article is to widen the audience of, and appreciation for, Differential Geometry by developing the notion of curve curvature in an intuitive manner using rudimentary geometry and a minimal number of equations.

Curvature is recognized as one of the most fundamental topics in Differential Geometry. In fact, M. Spivak devotes an entire volume [?] (vol. II.) to the study of curvature within a historical framework. This second volume is a wonderful, modern treatment of the notion of curvature as it evolved from the original investigations of L. Euler to later extensions by C. F. Gauss and B. Riemann. In the text, [?] Differential Geometry is approached from the viewpoint of E. Cartan. Aiming to present notions of Differential Geometry to a wider audience with less mathematical background, B. O’Neill states “This book is an elementary account of geometry of curves and surfaces. It is written for students who have completed standard courses in calculus and linear algebra, and its aim is to introduce some of the main ideas of differential geometry” ([?] page ix). This paper seeks to further simplify the topic of curvature by making it accessible to students who possess a rudimentary understanding of only geometry and algebra. Considering only lines, circles, and ratios, we present an intuitive, geometric understanding of this subject. It is our hope that students may be interested in continuing on to a more rigorous treatment of the subject in the future.

Without reading advanced texts, everyone has an intuitive understanding of curvature; shapes like straight lines (___) have no curvature while shapes like the curve ( ~ ) are curved. The idea of curvature is even built into the names of these shapes (straight and curved). Delving deeper, one sees the challenge is not so
Figure 3: The Normal Vector. The vector $N$ predicts the direction in which the vector $T$ will turn. Note that at inflection points, denoted by $\bigcirc$, the normal vector abruptly changes direction.

Notice that during the first bend in the curve in Figure ??, the vector $T$ is turning down as the magnetic ball moves from left to right, and, correspondingly, the normal vector $N$ is pointing down. In the next bend, the vector $T$ turns up and so the normal vector $N$ points up. There are points on a curve where the normal vector is not strictly defined. At these points, denoted by $\bigcirc$ in Figure ?? and called inflection points, there are two vectors perpendicular to $T$ and the choice for $N$ can therefore be made by simply flipping a coin. That is, the vectors $N$ will abruptly change direction from pointing down/up to up/down at these points of inflection.

3 Curves of Constant Curvature or Turning (Lines and Circles)

Now that we know how $T$ and $N$ work together to indicate how a plane curve turns, we can say that:

At each point, curve curvature quantifies the degree of deviation of the curve from a straight line.

The change in $T$ is used to precisely quantify this deviation. Intuitively, if a curve has a sharp turn and so $T$ is changing rapidly, then the curvature should have a large value. In contrast, if a turn is gradual, and so $T$ is changing slowly, then curvature should be small. Note that since $T$ and $N$ change together we can quantify these deviations using $N$ just as well as $T$. When discussing lines, studying the change in $T$ natural, while in the case of circles, $N$ is a better choice.

When a curve has a constant curvature value from one location to another, one can imagine the curve as being constructed from a series of uniformly bent components. The first example of a curve of constant curvature is a line. The line (______) can be viewed as an object comprised of a series of the smaller straight-line-components (---) each with no curvature. Notice that in Figure ?? the direction vector $T$ does not turn. Since $T$ does not change, the curvature should have a constant value 0. Recall that when $T$ does not change, $N$ is not uniquely determined. In Figure ?? we made a choice of $N$ pointing up; but $N$ pointing down could have been equally valid.

A circle is another example of a curve with constant curvature. Like a toy train set which has been
4 The Osculating Circle of a Plane Curve and Curve Curvature

Equation (??) relates the curvature of a circle to its radius, and is the key to developing a tool that can be applied to curves of non-constant curvature. This tool is called the osculating circle.

Except at points where the curve is perfectly flat, there is a circle of some radius that osculates (which means, kisses) the curve (see Figure ??). The osculating circle can be viewed as the best circular approximation of the curve at that instant. This means that this circle is of the perfect radius so as to conform to the shape of the curve at that point. As shown in Figure ??, this circle’s radius can change based on the location it kisses. That is, for the circle to conform to the contours of the curve it must be able to change its shape (and therefore, its radius) accordingly. In fact, Figure ?? shows a tacit relationship between the radius of the osculating circle and the curvature of the curve. At locations where the curve has a large curvature, for example at the point where the left-most circle kisses the curve, the radius of the osculating circle is small. The opposite is true at the point where the right most circle kisses the curve; the curve is flatter and so the radius of the osculating circle is larger. The middle circle kisses a point where the curve appears to be nearly flat (almost line-like), and therefore the radius of the osculating circle is much larger. These observations strongly suggest that we define the curvature of a curve at a given point as simply the reciprocal of the radius of the osculating circle at that point.

At a point where the curve is perfectly flat, we define the curvature to be 0. If we try to draw an osculating circle at such a point, we can never make the radius large enough for the circle to conform to the curve. For the circle to conform, it would have to have an infinite radius or, in other words, the best approximation at such a point is a line and not a circle. As its radius grows, a circle limits to a line. Likewise, the reciprocal of its radius limits to 0.

Think back to the curve you created by bending your long piece of straight wire. Your curve will most likely not have constant curvature since you will have decided to make a more interesting shape than a line or a circle. And while your curve, on the whole, is not a line or a circle, you can think of your curve as being constructed from a series of various sized straight-line segments and circular bends. At flat points the curvature is defined to be 0, while at any other point the curvature of the curve is defined to be the curvature of its osculating circle. That is, as a mathematical formula, we define the curvature $\kappa_\odot$ of generic curve as $0$ at flat points and

$$\kappa_\odot = \frac{1}{r_\odot} \text{ where } r_\odot \text{ is the radius of the osculating circle.} \quad (2)$$
7 Example: The Archemedean Spiral

As a final example, we analyze the Archemedean spiral using the same techniques as in the previous example.

(a) The Archemedean Spiral.

(b) Osculating Circles of Archemedean Spiral Are Decreasing in Radius

(c) Osculating Circles of Fresnel Spiral Are Also Decreasing in Radius

Figure 13: Archemedean Spiral vs. Fresnel Spiral. Do the radii of the osculating circles decrease in the same way?

Figure 14: Curvature Values at $A$, $B$, and $C$. Finding osculating circles at these three points and computing their reciprocals gives $A$ ($\kappa_\odot = 0.51$), $B$ ($\kappa_\odot = 0.63$), and $C$ ($\kappa_\odot = 1.56$).

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Figure 15: Archemedean Spiral Data The curvature of this spiral grows non-linearly with time.
We now switch into visual interactive mode.

8 2D Plane Curves Visual Interactive

Here is what we know about 2D Plane Curves:
There is no reason these results cannot be extended to 3D Space Curves.

The vector T points in the direction of the curve.
The vector N is perpendicular to T.

Osculating or “Kissing” Circles are the best circular approximation of the curve at a point.

The Fresnel spiral has osculating circles of smaller and smaller radius...

...so the curvature gets larger and larger.

This is the Fresnel spiral. Its curvature increases linearly with time on the curve.
The radius of osculating circle is about R=3/2 and the curvature is about 1/R=2/3

The Sine curve is the most curved it can be here.

Move the point just a bit more and the radius gets bigger so the Sine curve gets flatter.

The Sine curve is the most curved it can be here.

The radius of this circle is R=1
The curve curvature is 1/R=1

The radius of this osculating circle is R=9/2
The curvature of the curve is 1/R=2/9

Osculating or “Kissing” Circles are the best circular approximation of the curve at a point.

The vector T points in the direction of the curve.
The vector N is perpendicular to T.

The Fresnel spiral has osculating circles of smaller and smaller radius...

...so the curvature gets larger and larger.

The Sine curve is the most curved it can be here.

The radius of this circle is R=1
The curve curvature is 1/R=1

The Sine curve is consecutively flatter at these points b/c the radius of the osculating circles are consecutively bigger
Here is what we can say in 3D. The osculating circle still works for a circular helix of constant curvature.

A 3D Circular Helix... with an osculating circle.

See. Except for perspective, the osculating circles are actually circles.

Each osculating circle is of the same radius, so the circular helix has constant curvature. The circular helix is the 3D equivalent of the 2D circle. They are both are constant curvature.